Problem Set 1

23.5.23

1) Prove the followings by induction or otherwise:

- $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}=\frac{n(n+2)}{n+1}$
- $1.1!+2.2!+3.3!+\cdots+n.n!=(n+1)!-1$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$
- $\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\dots\frac{2n-1}{2n}\leq\frac{1}{\sqrt{2n+1}}$
- $\frac{1}{2}.\frac{3}{4}.\frac{5}{6}\dots\frac{2n-1}{2n}\leq\frac{1}{\sqrt{3n+1}}$
- 3^n has tenth digit even for all natural numbers n.

2) Let $\{u_n\}_{n\geq1}$ be a sequence of real numbers defined as $u_1=1$ and $u_{n+1}=u_n+\frac{1}{u_n}$ for all $n\geq1$. Prove that $u_n\leq\frac{3\sqrt{n}}{2}$ for all n.

3) Rahul and Rohit just became friends with Neha and they want to know when her birthday is. Neha gives them a list of possible dates,

May 15|May 16|June 17|June 18|July 14|July 16|July 19|Aug 14|Aug 15|Aug 17

Neha then tells Rahul and Rohit separately the month and date of her birthday respectively.

Rahul: I don't know when Neha's birthday is, but I know that Rohit doesn't know also.

Rohit: At first, I don't know when Neha's birthday is, but I know now.

Rahul: Then I also know when it is.

So, when is Neha's birthday?

4) A series is formed in the following manner:
$A(1) = 1$; and $A(n) = f(m)$ number of $f(m)$ followed by $f(m)$ number of 0; where m is the number of digits in $A(n-1)$ and $f(m)$ is the remainder when m is divided by 9. Find sum of digits of $A(30)$.

Problem Set 2

30.5.23

1)Prove that 3^n has tenth digit even for all natural numbers n.

[Hint: Think about the cyclicity of 3]

2) Find all possible $n, k \in \mathbb{N}$, such that

$$1! + 2! + 3! + 4! + \cdots + n! = k^2$$

[Hint: Check for $n = 1,2,3,4$ etc. then for the next numbers, try to conclude something from the last digit of $1! + 2! + .. + n!$]

3)Prove that $5|3^{2008} + 4^{2009}$

4)$K = 1^{3017} + 2^{3017} + \cdots + 200^{3017}$, prove that $201|K$. Then prove that $200|K$. Can you conclude $40200|K$? If yes, why?

Problem Set 3

6.6.23

1) $\sqrt{\log_{\mathrm{b}} n} = \log_b \sqrt{n}$ and $b \log_b n = \log_b bn$, then the value of n is equal to $\frac{j}{k}$, where j and k are coprime. What is $j + k =$?

2) $d = \gcd(n^2 + 20, (n+1)^2 + 20)\,, n \in \mathbb{N}$. Prove that $d|81$.

3) $d_n = \gcd(n^3 + n^2 + 1, n^3 + n + 1)\,, n \in \mathbb{N}$, find $d_{3^{2019}}$.

4) In a page, there are 10 statements. First statement says, exactly 1 of the 10 statements is false. Second statement says, exactly 2 is false….and so on. The tenth statement says that all are false. Prove that-

a) Exactly one statement is correct.

b) Hence find which one is correct.

5) Five sailors survive a shipwreck and swim to a tiny island where there is nothing but a coconut tree and a monkey. The sailors gather all the coconuts and put them in a big pile under the tree. Exhausted, they agree to go to wait until the next morning to divide up the coconuts.

At 1 o'clock, the 1st sailor divides the coconuts into 5 piles, but 1 coconut is left over. He gives that to the monkey, hides his coconuts and puts the rest of the coconuts under the tree.

At 2 o'clock, the 2nd sailor wakes up. Not realising that the 1st sailor has already taken his share, he too divides the coconuts up into 5 piles, leaving one coconut over, which he gives to the monkey. He then hides his share and puts the remainder back under the tree. At 3,4,5 o'clock, the same process is carried on. In the morning, all wake up and try to look innocent. None makes a remark about the diminished pile of coconuts and none decides to be honest and admit that they have already taken their share. Instead, they divide the pile up into 5 piles, for the 6th time, and find there is yet again one over

What is the smallest number of coconuts that there could have been in the original pile?

Problem Set 4

13.6.23

1) Prove that $n^5 + 4n$ is divisible by 5 for any integer n.

2)$p, q, r \in \mathbb{N}$, all greater than 3, such that $q - p = r - q$. Prove that $q - p$ is divisible by 6.

3) $p_1 = 2, p_2 = 3, p_3 = largest\ prime\ factor\ of\ (p_1 p_2 + 1) = 7, p_4 = largest\ prime\ factor\ of\ (p_1 p_2 p_3 + 1) = 43$. In general, $p_n = largest\ prime\ factor\ of (p_1 p_2 \dots . p_{n-1} + 1) \forall n \geq 3. Prove\ that\ p_n \neq 5\ \forall n \in \mathbb{N}$

4) I.S.I. entrance U.G.A paper consists of 30 questions. The marks are awarded in such a way that if a person gets a question correct, he gets +4 marks, but if he does it wrong, he gets 0 marks and if doesn't attend the question he gets +1 marks. Find all possible numbers that a student can get in the U.G.A paper.

Problem Set 5

20.6.23

1) $a, b, c \in \mathbb{N}, 6|(a + b + c)$, prove that $6|a^3 + b^3 + c^3$

2) The natural numbers x, y and z satisfy the equation $x^2 + y^2 = z^2$. Prove that at least one of them is divisible by 3.

3) Let a, b, c be positive real numbers such that: $ab - c = 3$

$abc = 18$ Calculate the numerical value of $\frac{ab}{c}$

4)

$$a_{n+1} = \left(\frac{n+1}{n-1}\right)\left(\sum_{i=2}^{n} a_i\right) \ \forall n \geq 2,$$

If $a_2 = 3, find\ a_{2023}$.

5) If $2 = p_1 < p_2 < \cdots < p_n$, where each p_i is prime, show that $p_1 p_2 \ldots p_n + 1$ can never be a perfect square.

6) There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that $\log_{20x}(22x) = \log_{2x}(202x)$. The value $\log_{20x}(22x)$ can be written as $\log_{10}(\frac{m}{n})$, where m and n are relatively prime positive integers. Find $m + n$.

Problem Set 6

27.6.23

1) Find the last digit of the number $7^{7^{7}}$.

2) Find the remainder when the number 3^{1989} is divided by 7.

3) Simplify: $\frac{\sqrt{45+\sqrt{1}}+\sqrt{45+\sqrt{2}}+\cdots\ldots\ldots\sqrt{45+\sqrt{2024}}}{\sqrt{45-\sqrt{1}}+\sqrt{45-\sqrt{2}}+\cdots\ldots\ldots\sqrt{45-\sqrt{2024}}}$

Problem Set 7

7.7.23

1) Let r be the root of equation $x^2 + 2x + 6$. The value of $(r + 2)(r + 3)(r + 4)(r + 5) = a$. Find $a + 200$.

2)Find the remainder when the number $1989.1990.1991 + 1992^3$ is divided by 7.

3)Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Problem Set 8

15.7.23

1) Given that $p, p + 10$ and $p + 14$ are prime numbers, find p.

2) Prove that there are no natural numbers a and b such that $a^2 - 3b^2 = 8$.

3) Prove that the sum of squares of 5 consecutive numbers can't be a perfect square.

Note: While dealing with problems about cubes of integers, it is often useful to analyze remainders modulo 7 or modulo 9. In either case there are only three possible remainders $\{0, 1, 6\}$ and $\{0, 1, 8\}$ respectively.

4) Given the pair of prime numbers p and $p^2 + 2$, prove that $p^3 + 2$ is also a prime number. [Hint: Think when can this happen!]

5) Find p such that $p, 4p^2 + 1, 6p^2 + 1$ all are prime.

6) Define $a = p^3 + p^2 + p + 11$ and $b = p^2 + 1$, where p is any prime number. Let $d = \gcd(a, b)$. Find the possible values of b.

7) Suppose $ax_0 + by_0 = d$, where $d = \gcd(a, b)$. Is (x_0, y_0) unique?

8) Find the remainder when 2^{1990} is divided by 1990.

9) Prove that $\frac{k^7}{7} + \frac{k^5}{5} + \frac{2k^3}{3} - \frac{k}{105} \in \mathbb{Z},\ \forall k \in \mathbb{N}$

10) If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$,then a, b, c, d four can't be odd simultaneously.

Problem Set 9

21.7.23

1) Suppose $ax_0 + by_0 = d$, where $d = \gcd(a, b)$. Is (x_0, y_0) unique?

2) Find the remainder when 2^{1990} is divided by 1990.

3) Prove that $\frac{k^7}{7} + \frac{k^5}{5} + \frac{2k^3}{3} - \frac{k}{105} \in \mathbb{Z},\ \forall k \in \mathbb{N}$

4) If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$,then a, b, c, d four can't be odd simultaneously.

5) k is a number which gives remainder 4 when divided by 9. Then find the number of solutions of $x^3 + y^3 + z^3 = k$.

6) Find the least possible value of a + b, where a, b are positive integers such that 11 divides a + 13b and 13 divides a + 11b.

7) $N = 13 \times 17 \times 41 \times 829 \times 56659712633$. It is known that N is a 18 digit number with 9 of the ten digits from 0 to 9 each appearing twice. Find the sum of the digits of N.

8) Let $a_1, a_2, a_3, \dots . a_n$ be positive integers which satisfies $a_1 < a_2 < a_3 < \cdots < a_n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \frac{1}{a_n} = 1$. Find a general method to find a tuple $(a_1, a_2, \dots . a_n)$ satisfying the above property.

Problem Set 10

8.8.23

1) Let a and b be natural numbers such that $a + b, a - 2b, 2a - b$ are all distinct squares. What can be the smallest possible value of b?

2) $\forall n \in \mathbb{N}, let\ s(n) =$ number of ordered pairs (x, y) of positive integers for which $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ Determine the set of positive integers for which $s(n) = 5$.

3) $A = \sum_{n=1}^{50}\left(x^n + \frac{1}{x^n}\right)\ and\ B = \sum_{n=1}^{10}\left(x^{5n-2} + \frac{1}{x^{5n-2}}\right), If\ x + \frac{1}{x} = 4, then\ find\ \frac{A}{B}$

Problem Set 11

29.8.23

1) Prove that there exists a natural number n such that $n+1, n+2, \ldots, n+2023$ all are composite.

2) Find the remainder when the number 3^{2023} is divided by 7.

3) Show that the quadratic equation $x^2 + 7x - 14(q^2+1) = 0 (q \in \mathbb{Z})$ has no integral root.

4) Suppose a, b are integers and a + b is a real root of $x^2 + ax + b = 0$, what is the maximum possible value of b^2?

5) Consider all non-empty subsets of the set {1,2,…,n}. For every such subset, we find the product of reciprocals of each of its elements. Denote the sum of all these products by S_n. For example, $S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{1.2.3}$

A) Show that $S_n = \frac{1}{n} + \left(1 + \frac{1}{n}\right) S_{n-1}$.

B) Prove using (A) that $S_n = n$.

Problem Set 12

5.9.23

1) Let the divisors of n be $1 = d_1 < d_2 < \cdots < d_k = n$. $\emptyset$ is euler's totient function and ϕ is null set. Let us define the following set $N_n = \{1,2,..,n\}$ and $S_i = \{x: x \in N_n \; and \; \gcd(x,n) = d_i\} \; \forall i = 1,2,\dots,k$.

i) Prove that S_i's are mutually disjoint, i.e., $S_i \cap S_j = \phi$, if $i \neq j$.

ii) Prove that, $|S_i| = \emptyset\left(\frac{n}{d_i}\right)$

iii) Prove that $\bigcup_{i=1}^{k} S_i = N_n$.

iv) Hence, prove that $\emptyset(d_1) + \emptyset(d_2) + \cdots + \emptyset(d_k) = n$.

2) Let n be a natural number. Let $\sigma(n)$ be the sum of divisors of n.

i) Prove that $\emptyset(x) \leq x$ for any natural number x.

ii) From the first question, we know that $\emptyset(d_1) + \emptyset(d_2) + \cdots + \emptyset(d_k) = n$. Now use the 1st part of question 2 to prove that $\emptyset(n) + \sigma(n) \geq 2n$

3) Determine all positive integers n such that the product of digits of n equals $n^2 - 15n - 27$.

4) Consider the series $50 + n^2$: 51,54,59,66,75 … … … If we take the greatest common divisor of 2 consecutive terms in the series, we obtain 3, 1, 1, 3………. What is the sum of all distinct elements in the 2nd series?

5) Let $n = 2^{31} \times 3^{19}$. How many divisors of n^2 are less than n but do not divide n.

6) Let p_1, p_2, p_3 be primes with $p_2 \neq p_3$, such that $4 + p_1p_2$ and $4 + p_1p_3$ are perfect squares. Find all possible values of p_1, p_2, p_3

Problem Set 13

26.9.23

Number Theory Compiled Problems

1) Integers a, b, c satisfy $a + b - c = 1, a^2 + b^2 - c^2 = -1$. Then what is sum of all possible distinct values of $(a^2 + b^2 + c^2)$?
2) Let a and b be natural numbers such that $a + b, a - 2b, 2a - b$ are all distinct squares. What can be the smallest possible value of b?
3) $Find\ the\ least\ possible\ value\ of\ a + b, where\ a, b\ are\ positive\ integers\ such\ that\ 11\ divides\ a + 13b\ and\ 13\ divides\ a + 11b.$
4) Let $f: \mathbb{N} \to \mathbb{N}$ such that $f(f(n))$ = no. of divisors of n. Prove that if p is a prime, then $f(p)$ is a prime. (**)
5) $p_1 = 2, p_2 = 3, p_3 = largest\ prime\ factor\ of\ (p_1 p_2 + 1) = 7, p_4 = largest\ prime\ factor\ of\ (p_1 p_2 p_3 + 1) = 43$. In general, $p_n = largest\ prime\ factor\ of (p_1 p_2 \dots . p_{n-1} + 1) \forall n \geq 3. Prove\ that\ p_n \neq 5\ \forall n \in \mathbb{N}$
6) Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0 (q \in \mathbb{Z})$ has no integral root.
7) $N = 13 \times 17 \times 41 \times 829 \times 56659712633$. It is known that N is a 18 digit number with 9 of the ten digits from 0 to 9 each appearing twice. Find the sum of the digits of N. [Hint: If n is a natural number, prove that $n - sum\ of\ digits\ of\ n$ is always divisible by 9.]
8) $n \in \mathbb{N}, n$ has k divisors. $d_1 < d_2 < \cdots < d_k, and\ d_5^2 + d_6^2 = 2n + 1$ Find $k\ and\ n$. (**)
9) Solve: $y^2 = x^3 + 7$, solve for x, y integers. (**)

10) $\forall n \in \mathbb{N}, let\ s(n) =$ number of ordered pairs (x, y) of positive integers for which $\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$ Determine the set of positive integers for which s(n)=5.

11) Prove that $2^n \nmid n!\ \forall n \in \mathbb{N}$

12) Determine the last three digits of $\sum_{n=2}^{10,000,000}(n^7 + n^5)$

13) $g(x) = (4a - 3d)x^5 + (4b - 3e)x^4 + (4c - 3f)x^3 + (4d - 3a)x^2 + (4e - 3b)x + (4f - 3c)$. We define the function g(x) as above where a, b, c, d, e, f are single digit natural numbers independent of x. If $g(10) = 0$, then find $a + b + c + d + e + f$. [Hint: Think in terms of numbers]

14) $2^m + 3^n = a^2$. Solve for m, n, a being natural numbers (**)

15) Let a, b, c, d be distinct digits such that product of the 2 digit numbers $ab\ and\ cd$ is of the form ddd. Find all possible values of $a + b + c + d$?

16) Prove that there exist 100 consecutive natural numbers, such that exactly 3 of them are primes. (*)

17) Consider the sequence $\{101, 10101, 1010101, 101010101, \ldots\}$. How many primes are present in the sequence?

18) $n \in \mathbb{N}, n\ has\ k\ divisors. 1 = d_1 < d_2 < \cdots \ldots . d_k = n. Given\ d_{13} + d_{14} + d_{15} = n. Find\ k.$

19) $A = \sum_{n=1}^{50}\left(x^n + \frac{1}{x^n}\right)\ and\ B = \sum_{n=1}^{10}\left(x^{5n-2} + \frac{1}{x^{5n-2}}\right), If\ x + \frac{1}{x} = 4, then\ find\ \frac{A}{B}$

20) Consider the series $50 + n^2$: 51,54,59,66,75 … … … If we take the greatest common divisor of 2 consecutive terms in the series, we

obtain 3, 1, 1, 3………. What is the sum of all distinct elements in the 2nd series?

21) A series is formed in the following manner:
$A(1) = 1;\ and\ A(n) = f(m)$ number of $f(m)$ followed by $f(m)$ number of 0; where m is the number of digits in $A(n-1)$ and $f(m)$ is the remainder when m is divided by 9. Find sum of digits of $A(30)$

22) $x\ \in \mathbb{R}^+, \left\lfloor x\left\lfloor x\lfloor x\lfloor x\rfloor\rfloor\right\rfloor\right\rfloor = 88, then\ \lfloor 7x\rfloor = ?$ (*)

23) Find the last non-zero digit of 2022! (*)

24) Consider the following number:
258145266804692077858261512663
It is the 13th power of some natural number. Find the number.

25) Let $a_1, a_2, \ldots \ldots . a_n$ be integers. Show that there exist integers k and r such that the sum $a_k + a_{k+1} + \cdots + a_{k+r}\ is\ divisible\ by\ n$

26) Let S be the set of all positive integers n such that n and n+1 have exactly four divisors and have their divisors add to the same value. Let m be the number of elements in S and b the sum of these m elements. Find $m\ and\ b$.

27) Let x_n denotes the n^{th} non square positive integer. Then $x_1 = 2, x_2 = 3, x_3 = 5, x_4 = 6\ etc$. For a positive real number x, denote the integer closest to it by <x> If x=m+0.5, where m is an integer, then <x>=m. Eg. <1.2> = 1, <2.8> = 3, <3.5> = 3. Show that $x_n = n + <\sqrt{n}>$

28) A corona sequence is an increasing sequence of 16 consecutive odd integers whose sum is a perfect cube. How many corona sequences are there with 3 digit numbers only?

29) Let P(x) be the polynomial when $(x+7)^{100}$ is divided by $(x^2 - x - 1)$. Now find the remainder when P(x) is divided by 11

30) Let $a_1, a_2, a_3, \dots . a_n$ be positive integers which satisfies $a_1 < a_2 < a_3 < \cdots < a_n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots \frac{1}{a_n} = 1$. Find a general method to find a tuple $(a_1, a_2, \dots . a_n)$ satisfying the above property.

31) Consider all non-empty subsets of the set $\{1,2, \dots, n\}$. For every such subset, we find the product of reciprocals of each of its elements. Denote the sum of all these products by S_n. For example, $S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{1.2.3}$

A) Show that $S_n = \frac{1}{n} + \left(1 + \frac{1}{n}\right) S_{n-1}$.

B) Prove using (A) that $S_n = n$.

C) Prove not using (A) that $S_n = n$. (*)

32) Find all positive integers n such that $5^n + 1$ is divisible by 7

33) Calculate $\left\{\frac{a}{p}\right\} + \left\{\frac{2a}{p}\right\} + \left\{\frac{3a}{p}\right\} + \cdots + \left\{\frac{(p-1)a}{p}\right\}$ where p is a prime and a is natural number such that p does not divide a.

34) $\sum_{i=1}^{n}\left(\left\lfloor\frac{n}{i}\right\rfloor - \left\lfloor\frac{n-1}{i}\right\rfloor\right) = A, \sum_{i=1}^{n}\left(\left\lfloor\frac{n}{i}\right\rfloor^2 - \left\lfloor\frac{n-1}{i}\right\rfloor^2\right) = B$, where n is a natural number. Prove that number of divisor is A, and sum of divisors of n is $\frac{A+B}{2}$

35) Find the three-digit positive integer $\underline{a}\,\underline{b}\,\underline{c}$ whose representation in base nine is $\underline{b}\,\underline{c}\,\underline{a}_{\text{nine}}$, where a, b, and c are (not necessarily distinct) digits.

36) $d = \gcd(n^2 + 20, (n+1)^2 + 20)\,, n \in \mathbb{N}$. Show that $d|81$

37) $d_n = \gcd(n^3 + n^2 + 1, n^3 + n + 1)\,, n \in \mathbb{N}$.Find $d_{3^{2022}}$.

38) Let p be a prime number bigger than 5. Suppose the decimal expansion of $\frac{1}{p}$ looks like $0.\overline{a_1 a_2 \dots a_r}$ where the line denotes a

recurring decimal. Prove that 10^r leaves a remainder of 1 on dividing by p.

39) Is a there any natural number n such that when $n!$ written in base ten will end with exactly 2022 zeros?

40) Prove that every positive rational number can be expressed uniquely as a finite sum of the form

$$a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

Where a_n are integers such that $0 \leq a_n \leq n-1\ \forall n > 1$ (**)

41) Let $n \geq 2$ be an integer. Let m be the largest integer which is less than or equal to n, and which is a power of 2. Put l_n= least common multiple of 1,2,…,n. Show that $\frac{l_n}{m}$ is odd, and that for every integer $k \leq n, k \neq m, \frac{l_n}{k}$ is even. Hence, prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer

42) For any positive integer n, and i = 1, 2, let $f_i(n)$ denote the number of divisors of n of the form 3k + i (including 1 and n). Define, for any positive integer n, $f(n) = f_1(n) - f_2(n)$.Find the value of $f(5^{2022}), f(21^{2022})$.

43) Find all the possible two-digit numbers ab such that $ab = 4(a! + b!)$, where ab is the number whose first digit is a and second digit is b.

44) If $\sqrt{\log_b n} = \log_b \sqrt{n}$ and $b \log_b n = \log_b bn$, then the value of n is equal to $\frac{j}{k}$, where j and k are relatively prime. What is $j + k$?

45) Find number of pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

Problem Set 14

31.10.23

1) Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_0$ be a polynomial of degree n with real coefficients $a_0, a_1, \dots \dots \dots a_n$ such that $a_n = 1$ and $a_i^2 = 1$ for i = 0,1,2,…….,(n-1). Suppose all the roots $c_1, c_2, c_3 \dots \dots \dots c_n$ of the equation P(x)=0 are integers. Find $c_1^2 + c_2^2 + \cdots c_n^2$. Hence find all such polynomials P(x).

2) Find all positive integers a, b such that each of the equations $x^2 - ax + b = 0 \; and \; x^2 - bx + a = 0$ has distinct positive integral roots.

3) f(x) is a degree 4 polynomial satisfy $f(n) = \frac{1}{n}$ for n=1,2,3,4,5. If $f(0) = \frac{a}{b}$,(a and b are co-prime positive integers), then a+b =?

4) Find the number of real solutions of the equation: (x-1)(x-3)………(x-2021)=(x-2)(x-4)………(x-2020)

5) Let a, b be the roots of the equation $x^2 - 10cx - 11d = 0 \; and \; those \; of \; x^2 - 10ax - 11b = 0 \; are \; c, d. Then \; a + b + c + d = ? \; (a \neq b \neq c \neq d)$

6) $x_1, x_2, \dots \dots \dots \dots x_n$ be complex numbers to satisfy the following set of equations

$x_1 + x_2 + \cdots \dots \dots x_n = n$
$x_1^2 + x_2^2 + \cdots \dots \dots x_n^2 = n$
$x_1^3 + x_2^3 + \cdots \dots \dots x_n^3 = n$
…..
……
$x_1^n + x_2^n + \cdots \dots \dots x_n^n = n$
Then, prove that $x_i = 1 \; \forall i = 1(1)n$

Problem Set 15
8.12.23

1)Prove the following version of rearrangement theorem: Let $a_1, a_2, \ldots\ldots a_n\ and\ b_1, b_2, \ldots\ldots b_n$ be 2 permutations of the numbers 1,2,……n. Show that

$$\sum_{i=1}^{n} i(n+1-i) \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} i^2$$

(Hint: Recall Cauchy-Scwartz inequality)

2)If $a, b, c > 0$ are sides of a triangle, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$$

3)$a, b, c \geq 0, a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$

(Hint: See what happens if $a^2+b^2+c^2 < abc,\ i.e.\ abc > a^2$)

4)$f(x) = \begin{cases} 0+\beta : \lfloor x \rfloor = 0\ mod\ 3 \\ 1+\beta : \lfloor x \rfloor = 1\ mod\ 3 \\ 2+\beta : \lfloor x \rfloor = 2\ mod\ 3 \end{cases}$ where $\lfloor x \rfloor$ denotes the greatest integer function. If $\sum_{n=1}^{\infty} \frac{f\left(3^n\sqrt{2023}\right)}{3^n} = 0$, then find value of β. (Hint: Write $\sqrt{2023}$ in powers of 3, i.e.,

$\sqrt{2023} = a_n 3^n + a_{n-1}3^{n-1} + \ldots + a_1.3 + a_0 + a_{-1}3^{-1} + a_{-2}3^{-2} + \ldots..$

Now, put into f and the summation and see what happens!)

Problem Set 16
15.12.23

1) Let a, b, c be positive reals such that $abc = 1$. Prove that

$$\frac{2}{(a+1)^2+b^2+1}+\frac{2}{(b+1)^2+c^2+1}+\frac{2}{(c+1)^2+a^2+1} \leq 1$$

2) $a, b, c \in \mathbb{R}^+, a+b+c = abc$. Prove that

$$\frac{1}{\sqrt{a^2+1}}+\frac{1}{\sqrt{b^2+1}}+\frac{1}{\sqrt{c^2+1}} \leq \frac{3}{2}$$

3) $$\begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$ Prove that the sum of entries of the table situated in different rows and different columns is not less than 1. [Hint: Consider the table with the entries reciprocal of this table]

4) Let $P(x), Q(x)$ be distinct polynomials with real coefficients such that the sum of the coefficients of each of the polynomials is s. If $P(x)^3 - Q(x)^3 = P(x^3) - Q(x^3)$, then prove that

- $P(x) - Q(x) = (x-1)^a r(x)$ for some integer $a \geq 1$ & a polynomial $r(x)$ with $r(1) \neq 0$.

- $s^2 = 3^{a-1}$, where a is as given in the previous.

5)Consider an acute angled triangle PQR such that C, I, O are circumcenter, incenter, orthocenter respectively. Suppose <QCR, <QIR, <QOR measured in degrees are α,β,γ.

Show that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} > \dfrac{1}{45}$.

Problem Set 17

27.12.23

1) If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c}+\frac{b^2+1}{c+a}+\frac{c^2+1}{a+b}\geq 3$$

2) What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}}\geq n(a+b+c)?$$

3) If $\dfrac{1-ix}{1+ix}=a-ib$ and $a^2+b^2=1$, where $a,b\in\mathbb{R}$ and $i=\sqrt{-1}$, then prove that $x=\dfrac{2b}{(1+a)^2+b^2}$

4) If $z=(3+4i)^6+(3-4i)^6$, then $Im(z)=?$

5) If $\dfrac{3}{2+cos\theta+isin\theta}=a+ib$, where $i=\sqrt{-1}$ and $a^2+b^2=\lambda a-3$, then $\lambda=?$

6) If $(x+iy)^{\frac{1}{3}}=a+ib$, then $\left(\dfrac{x}{a}+\dfrac{y}{b}\right)$ = ?

7) Try all the problems of Session 2 Exercise of Skills in Mathematics for JEE M & A, Algebra, Dr. SK Goyal, Arrant and come with doubts (but try your best first!!)

Problem Set 18
03.01.24

1) If $|z_1| = 1, |z_2| = 2, |z_3| = 3, |9z_1z_2 + 4z_3z_1 + z_2z_3| = 6$. find the value of $|z_1 + z_2 + z_3|$

2) z_1 & z_2 are two complex numbers, such that $\dfrac{z_1 - 2z_2}{2 - z_1\bar{z_2}}$ is uni modular, while $|z_2| \neq 1$, find $|z_1|$.

3) Solve: $z^2 + |z| = 0$

4) If the polynomial $7x^3 + ax + b$ is divisible by $x^2 - x + 1$, find the value of $2a + b$.

5) Find the real part of $(1 - i)^{-i}$

6) If $0 < amp(z) < \pi$, then $amp(z) - amp(-z) = ?$

7) If $|z_1| = |z_2|$ and $amp(z_1) + amp(z_2) = 0$, then

$a) z_1 = z_2$

$b) \bar{z_1} = z_2$

$c) z_1 + z_2 = 0$

$d) \bar{z_1} = \bar{z_2}$

8) Find the number of solutions of the equation $z^2 + \bar{z} = 0$

9) Find the simplest possible form of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$...upto 2n factors.

10) If $\theta \in \mathbb{R}$, then $\left(\dfrac{1 + sin\theta + icos\theta}{1 + sin\theta - icos\theta}\right)^n$ = ? In simplest possible form.

11) If α, β, γ are the cube roots of $p (p < 0)$, then for any x, y, z, $\dfrac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = ?$

12) If $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ are the n, nth roots of unity, then find the value of $\displaystyle\sum_{i=0}^{n-1} \frac{\alpha_i}{2023 - \alpha_i}$.

13) If $n \geq 3$, and $1, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are the n, nth roots of unity, then find the value of $\sum\sum_{1 \leq i < j \leq n-1} \alpha_i \alpha_j$

14) Find the value of $\sum_{k=1}^{10} [\sin\left(\frac{2k\pi}{11}\right) - i\cos\left(\frac{2k\pi}{11}\right)]$

15) TRY also the Session 4 (whatever covered) in Arihant book.

Problem Set 19
16.01.24

1) If a, b, c are distinct integers, then the minimum value of $|a + bw + cw^2| + |a + bw^2 + cw|$=?

2) Prove that the condition $(a\bar{b} + \bar{a}b)(b\bar{c} + \bar{b}c) + (c\bar{a} - \bar{c}a)^2 = 0$ implies that the equation $az^2 + bz + c = 0$ has one purely imaginary root.

3)

Let $A = \left\{ \frac{1967 + 1686i\sin\theta}{7 - 3i\cos\theta} : \theta \in R \right\}$. If A contains exactly one positive integer n, then the value of n is

4)

Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be nonzero.
Match each entry in **List-I** to the correct entries in **List-II.**

	List – I		**List – II**
(P)	$\|z\|^2$ is equal to	(1)	12
(Q)	$\|z - \bar{z}\|^2$ is equal to	(2)	4
(R)	$\|z\|^2 + \|z + \bar{z}\|^2$ is equal to	(3)	8
(S)	$\|z + 1\|^2$ is equal to	(4)	10
		(5)	7

The correct option is:
(A) (P) → (1) (Q) → (3) (R) → (5) (S) → (4)
(B) (P) → (2) (Q) → (1) (R) → (3) (S) → (5)
(C) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)
(D) (P) → (2) (Q) → (3) (R) → (5) (S) → (4)

5) If α and β are the roots of $z + \frac{1}{z} = 2(\cos\theta + \sin\theta)$, where $0 < \theta < \pi$, show that $|\alpha - i| = |\beta - i| = \sqrt{2}$

Problem Set 20

Srijan Chattopadhyay

January 30, 2024

1. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on the circles $(x-x_0)^2+(y-y_0)^2 = r^2$ and $(x-x_0)^2+(y-y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| = ?$
2. If $|z| \geq 3$, then determine the least value of $|z + \frac{1}{z}|$.
3. If $|z + \frac{4}{z}| = 2$, then find the minimum and maximum values of $|z|$.
4. For any integer k, let $\alpha_k = \cos(\frac{k\pi}{7}) + i\sin(\frac{k\pi}{7})$. Then find the value of $\frac{\sum_{i=1}^{12} |\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1}-\alpha_{4k-2}|}$.
5. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\}$. If $z = x + iy$ and $z \in S$, find locus of z.
6. Let z be a complex number satisfying $|z|^3+2z^2+4\bar{z}–8 = 0$, where$\bar{z}$ denotes the complex conjugate of z . Let the imaginary part of z be nonzero. $|z|^2$=? $|z - \bar{z}|^2 = ?|z|^2 + |z + \bar{z}|^2 = ?$ $|z + 1|^2 = ?$
7. This question paper Q no 13 and 14 in mathematics section.
8. This question paper Q no 04 in mathematics section.

Problem Set 21

Srijan Chattopadhyay

February 5, 2024

1. $z_1, z_2, \cdots$ be a sequence of complex numbers defined by $z_1 = i$ and $z_{n+1} = z_n^2 + i$ for $n \geq 1$. Then $|z_{2024} - z_1| = ?$

2. Consider the two subsets of $\mathbb{C}$, $A = \{\frac{1}{z} : |z| = 2\}$ and $B = \{\frac{1}{z} : |z-1| = 2\}$. Identify A and B as geometric figures.

3. Find the minimum possible value of $|z|^2 + |z-3|^2 + |z-6i|^2$, where z is a complex number.

4. Let $0 < a_1 < a_2 < \cdots < a_n$ be real numbers. Show that the equation $\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = 2015$ has exactly n real roots.

5. Find the values of x, y for which $x^2 + y^2$ takes the minimum value where $(x-5)^2 + (y-12)^2 = 14^2$.

6. Page 68 of arihant

Problem Set 22

Srijan Chattopadhyay

March 12, 2024

1. Draw graphs of $\frac{1}{x^{2n-1}}$, $\frac{1}{x^{2n}}$, $x^{\frac{1}{2n}}$, $x^{\frac{1}{2n-1}}$ for $n \in \mathbb{N}$.
2. Draw graphs of x and $[x]$, i.e. fractional and integer part of x. Also, ceiling of x.
3. Draw graphs of $[f(x)]$ assuming some quadratic function f. Can you generalize this in a visual way?
4. Draw graphs of $\sin x$, $\cos x$, $\tan x$, $cosecx$, $secx$, $cotx$. Also, draw the graphs of their inverse functions.
5. Draw graphs of 2^x, e^{-x}, $e^{-|x|}$ and $(1/2)^x$.
6. Draw graphs of $\frac{|\sin x|}{\sin x}$.

Problem Set 23

Srijan Chattopadhyay

April 1, 2024

1. Let A_k (respectively B_k) be the set of all k-tuples $(a_1, a_2, \cdots, a_k)$ of integers such that $0 \leq a_i \leq 4i$ and $\sum_{i=1}^{k} a_i$ is even (respectively odd). Find $|A_k| - |B_k|$ where $|S|$ denotes the cardinality of the set S.

2. On a coordinate plane, there are two regions M and N. M is confined by $y \geq 0, y \leq x$ and $y \leq 2 - x$ and N is determined by the inequalities $t \leq x \leq t + 1, 0 \leq t \leq 1$. Find the maximum common area of M and N.

3. Find number of distinct integers in the sequence $[\frac{1^2}{2024}], [\frac{2^2}{2024}], [\frac{3^2}{2024}], \cdots, [\frac{2024^2}{2024}]$.

4. Find number of real roots of $f(f(x)) = 0$, where $f(x) = x^3 - 3x + 1$.

5. Find $[\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}]$.

6. A hyperbola H with equation $xy = n$ (where $n \leq 1000$) is rotated $45°$ to obtain the hyperbola H'. Let the positive difference between the number of lattice points on H and H' be D. Given that both H and H' have at least one lattice point. Find the maximum possible value of D.

7. Let $\{x_n\}$ be a sequence such that $x_1 = 2, x_2 = 1$ and $2x_n - 3x_{n-1} + x_{n-2} = 0$ for $n > 2$. Find an expression for x_n.

8. In how many ways can a man wear 5 rings on the fingers (except the thumb) of one hand? (All the 5 rings should be utilized).

9. There are n straight lines in a plane, no two of which are parallel and no three are concurrent. Their point of intesections are joined. Prove that the number of fresh straight lines introduced is $\frac{n(n-1)(n-2)(n-3)}{8}$.

10. A closet has 5 pair of shoes. Find number of ways 4 shoes can be chosen from it so that there will be no complete pair.

11. Find last non zero digit of 2024!

12. $P(x) \in \mathbb{Z}(x)$. a, b, c are distinct integers such that $P(a) = b, P(b) = c, P(c) = a$. Prove that P can't exist.

13. Souradip and Soumalya go to a dinner party with 4 other couples, each person there shakes hand with everyone he/she doesn't know. Later, Souradip does a survey and discovers that everyone of the nine other attendees shook hands with a different number of people. How many people did Soumalya shake hannds with?

14. Suppose X is a finite set of points on the plane, not all on one line. Prove that there is a line passing through exactly 2 points of X.

Problem Set 24

Srijan Chattopadhyay

April 9, 2024

1. Solve the inequality: $\binom{x-1}{4} - \binom{x-1}{3} - \frac{5}{4}\binom{x-1}{2} < 0, x \in \mathbb{N}$.
2. Prove that $\sum_{k=0}^{9} x^k$ divides $\sum_{k=0}^{9} x^{kkkk}$.
3. Let, $S = \{1, 2, \cdots, 100\}$. For every non-empty subset A of S, let $m(A)$ denote the maximum element of A. Then, find $\sum m(A)$, where the summation is taken over al non-empty subsets A of S.
4. For any positive integer n, let $< n >$ denote the integer nearest to $\sqrt{n}$. Find $\sum_{n=1}^{\infty} \frac{2^{<n>}+2^{-<n>}}{2^n}$.
5. S is a finite subset of the set of positive integers. It has the property that whatever i, j belong to $S(i \neq j)$, then $\frac{i+j}{gcd(i,j)}$ also belong to S. Find all such subsets S.
6. Consider the smallest number in each of the $\binom{n}{r}$ subsets (of size r) of $S = \{1, 2, \cdots, n\}$. Show that the arithmetic mean of the numbers so obtained is $\frac{n+1}{r+1}$.
7. On an island live 13 purple, 15 yellow and 17 maroon chameleons. When two chameleons of different colors meet, they both change into the third color. Is there a sequence of pairwise meetings after which all chameleons have the same color?
8. Calculate $\sum_{r=0}^{47} \binom{50}{r}\binom{50-r}{3} 2^r$.
9. If n is prime greater than 3, then $(1 + 2\cos\frac{2\pi}{n})(1 + 2\cos\frac{4\pi}{n})(1 + 2\cos\frac{6\pi}{n}) \cdots (1 + 2\cos\frac{2n\pi}{n})$
10. Find the minimum value of $\sqrt{x^2 + y^2 - xy} + \sqrt{(1-x)^2 + (2-y)^2 - (1-x)(2-y)}$, where $0 < x < y$.
11. $x_1^2 + px_1 + q = x_2, x_2^2 + px_2 + q = x_3, x_3^2 + px_3 + q = x_1$. Let p, q be real numbers with $\alpha < \beta$ be the roots of the equation $x^2 + (p-1)x + q = 0$. What is the maximum number of solutions of the system of the equations above where $x_1, x_2, x_3 \in [\alpha, \beta]$ is?
12. You and your friend are playing a game. The game goes as follows: At the start, you are given a positive number N. You choose a positive factor n of N, where $n \neq N$. You compute $N - n$ and give the result to your friend. Your friend repeats the procedure with the integer she is given and gives the result to you, and so forth. For example, if you are given $N = 12$, you can choose $n = 3$ and give $N - n = 9$ to your friend. Now $N = 9$. Play continues in this fashion until one person is given $N = 1$ at which point the person (who is given $N = 1$) loses. Given that you and your friend will both play optimally, how many positive integers $N \leq 2024$ are there such that, when given at the start you will always win?
13. Consider all the numbers of the form $1 \pm \sqrt{2} \pm \sqrt{3} \pm \cdots \pm \sqrt{2024}$. Prove that when all of the numbers are multiplied, the product will belong to $\mathbb{Z}$.
14. $w, x, y, z \in \mathbb{N}, w^2 + x^2 + y^2 + z^2 = wxy + xyz + wxz + wyz$. Prove that there exists a solution (w_0, x_0, y_0, z_0) such that each of them is greater than 2024^{2024}.
15. Evaluate: $\sum_{m=1}^{n} \sum_{k=1}^{m} \sum_{p=k}^{m} \binom{n}{m}\binom{m}{p}\binom{p}{k}$

Problem Set 26

Srijan Chattopadhyay

May 22, 2024

1. "Run" is defined as a consecutive sequence of symbols till another symbol comes. For example, in AAAAABBBBAA, there are 2 runs of A and 1 run of B. In AAABBBBAAAAABBBBA, there are 3 runs of A and 2 runs of B. Now, assume there are 2 symbols A and B. R_1 = number of runs of A and R_2 = number of runs of B. Let $R = R_1 + R_2$ be the total number of runs. n_1 = total number of symbols A, n_2 = total number of symbols of B and $n = n_1 + n_2$. Let $R_{ij}, i = 1, 2, j = 1, 2, \cdots, n_i$ denote respectively the number of runs of type i, which are of length j. So, $\sum_{j=1}^{n_i} j.R_{ij} = n_i, \sum_{j=1}^{n_i} R_{ij} = R_i, i = 1, 2$ For example, in AABBAAB, $R_1 = R_2 = 2$, $n_1 = 4, n_2 = 3, n = 7, R = 4$. Now assume that you have observed a totally random sequence with n_1 many A and n_2 many B. Prove the followings:

   (a) $r_1 = r_2$ or $r_1 = r_2 \pm 1$.

   (b) $P(R_1 = r_1, R_2 = r_2) = \frac{c\binom{n_1-1}{r_1-1}\binom{n_2-1}{r_2-1}}{\binom{n_1+n_2}{n_1}}$, where $c = 2$ if $r_1 = r_2$ and $c = 1$ if $r_1 = r_2 \pm 1$.

   (c) $P(R = 2d) = \frac{2\binom{n_1-1}{d-1}\binom{n_2-1}{d-1}}{\binom{n_1+n_2}{n_1}}$

   (d) $P(R = 2d+1) = \frac{\binom{n_1-1}{d-1}\binom{n_2-1}{d}+\binom{n_1-1}{d}\binom{n_2-1}{d-1}}{\binom{n_1+n_2}{n_1}}$

   (e) $P(R_1 = r_1) = \frac{\binom{n_1-1}{r_1-1}\binom{n_2+1}{r_1}}{\binom{n_1+n_2}{n_1}}$

   (f) Probability that the r_1 runs of n_1 objects of A and r_2 runs of n_2 objects of B consists of exactly $r_{1j}, j = 1, \cdots, n_1$ and $r_{2j}, j = 1, \cdots, n_2$ runs of length j respectively is

   $$\frac{cr_1!r_2!}{\binom{n_1+n_2}{n_1}\prod_{i=1}^{2}\prod_{j=1}^{n_i} r_{ij}!}$$

   where $c = 2$ if $r_1 = r_2$ and $c = 1$ if $r_1 = r_2 \pm 1$.

   (g) $P(R_{1j} = r_{ij}, j = 1, \cdots, n_1) = \frac{\binom{n_2+1}{r_1}r_1!}{\binom{n_1+n_2}{n_1}\prod_{j=1}^{n_1} r_{ij}!}$

   (h) Now, let K = length of longest run of A. Take $n_1 = 5, n_2 = 6$. Find $P(K \le 5)$.

   (i) In general,

   $$P(K = k) = \sum_{r_1} \sum_{r_{11},\cdots,r_{1k}}^{r_{11}+\cdots,r_{1k}=r_1} \frac{r_1!\binom{n_2+1}{r_1}}{\binom{n_1+n_2}{n_1}\prod_{j=1}^{n_i} r_{ij}!}$$

   where, $\sum_{j=1}^{k} r_{ij} = r_1, \sum_{j=1}^{k} jr_{1j} = n_1, r_{1k} \ge 1, r_1 \le n_1 - k + 1, r_1 \le n_2 + 1$

2. Let $A_1, \cdots, A_m, B_1, \cdots, B_m$ be finite subsets of $\mathbb{N}$, such that $A_i \cap B_i = \emptyset, A_i \cap B_j \ne \emptyset$ for all $i \ne j$. Consider

   $$\Omega = (\cup_{i=1}^m A_i) \cup (\cup_{i=1}^m B_i)$$

   Now, let us consider a random permutation of Ω. Let E_i be the event that elements of A_i comes before elements of B_i. Now, prove the followings:

   (a) $P(E_i) = \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}$

   (b) Prove that atmost one of E_i can happen.

(c) Hence prove that,

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

(d) If $|A_i| = k, |B_i| = l$, $m \leq \binom{k+l}{l}$

Problem Set 27

Srijan Chattopadhyay

May 21, 2024

1. Let 18 equally spaced points be chosen on the circumference of a circle. How many ways can these points be connected in pairs such that the resulting 9 segments do not intersect?

2. $a_1, a_2, \cdots, a_n \in \mathbb{N}, a_i > 1 \forall i = 1, \cdots, n$. $\sum_{i=1}^{n} \frac{1}{a_i} = K \in \mathbb{N}, M = a_1 a_2 \cdots a_n, P(x) = M(x+1)^K - (x+a_1)(x+a_2)\cdots(x+a_n)$. Degree of $P(x)$ is $n[n > K]$. Prove that P has no positive roots.

3. 10 candidates participate for olympiad, which is organized around a table. There are 5 versions of the test and each candidate will receive exactly 1 version. No 2 consecutive candidates will get same version. How many ways there are to give the questions?

4. $f(x) = k + \beta$ when $[x] = k \bmod 3$, for $k = 0, 1, 2$, where $[x]$ means the greatest integer function of x. If $\sum_{n=1}^{\infty} \frac{f(3^n \sqrt{2024})}{3^n} = 0$, then find the value of β.

5. Prove that for any prime $p \geq 3$, the number $\binom{2p-1}{p-1} - 1$ is divisible by p^2.

6. If z_1, z_2, z_3 are non zero complex numbers such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, then prove that z_1, z_2, z_3 lie on a circle passing through the origin.

7. Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$ where $a, b, c \in \mathbb{Z}$ and $c \neq 0$. Suppose $f(1) = 0$ and the roots of $g(x) = 0$ are squares of the roots of $f(x) = 0$. Find the value of $a^{2024} + b^{2024} + c^{2024}$.

8. If $x_1, x_2, \ldots, x_{2024}$ are the zeros of $P(x) = x^{2024} + 2024x - 1$, find the value of $\prod_{i=1}^{2024} \frac{x_i}{x_i - 1}$.

9. Let a, b, c be roots of the polynomial $P(x) = x^3 - 4x^2 + 4x + 5$. Determine the value of
$$\frac{1-a^3}{b+c} + \frac{1-b^3}{c+a} + \frac{1-c^3}{a+b}$$
.

10. Let a_1, a_2, a_3, a_4 be real numbers satisfying $a_1 + a_2 + a_3 + a_4 = 1$. For $k = 1, 2, 3, 4$, the following condition holds:
$$\sum_{i=1}^{4} \frac{a_i}{k^2 + i} = \frac{1}{k^2}$$
Find the value of $\frac{a_1}{26} + \frac{a_2}{27} + \frac{a_3}{28} + \frac{a_4}{29}$.

11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2) - f(y^2) = (x+y)(f(x) - f(y))$$
holds for all $x, y \in \mathbb{R}$.

Problem Set 28

Srijan Chattopadhyay

June 11, 2024

1. $x \in \mathbb{R}^+$ If $\lfloor x \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor \rfloor = 88$, $\lfloor 7x \rfloor = ?$

2. Consider the following number : 2581452668046920778582615126 63. It is the 13 th power of some natural number. Find the number.

3. Determine the number of functions f such that $f : \{1, 2, \cdots, 1999\} \to \{2000, 2001, 2002, 2003\}$ and satisfies the condition $f(1) + f(2) + \cdots + f(1999)$ is odd.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \le (x - y)^2 \ \forall x, y \in \mathbb{R}$.

5. An ant is moving in a co-ordinate plane in the following manner : it starts at $(6, 0)$ and in each step, it rotates by 60° anticlockwise about the origin and moves 7 units towards positive X direction. If (p, q) be the position of the ant after 2024 moves, find $p^2 + q^2$.

6. A positive integer is called CORONA if its digits are non-decreasing from left to right (for example, 1123445) is CORONA. Prove that $\forall n \in \mathbb{N}, \exists$ an n digit natural number which is CORONA and perfect square also.

7. Let A be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(xy) = xf(y)$ for all $x, y \in \mathbb{R}$. If $f \in A$, then show that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. For $g, h \in A$, define a function $g \circ h(x) = g(h(x))$ for all $x \in \mathbb{R}$. Prove that $g \circ h$ is in A and equal to $h \circ g$.

8. $a_1, a_2, \cdots, a_n$ are real numbers either $+1$ or -1. Prove that

$$2 \sin \left(a_1 + \frac{a_1 a_2}{2} + \frac{a_1 a_2 a_3}{2^2} + \cdots + \frac{a_1 a_2 \cdots a_n}{2^{n-1}} \right) \frac{\pi}{4} = a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{\cdots + a_n \sqrt{2}}}}$$

9. Try the problems listed here.

Problem Set 29

Srijan Chattopadhyay

August 18, 2024

1. Start exploring the **Calculus book** of the **Arihant JEE series**, look at the chapter named **"Function"**, **"Limit"** etc and solve the **sessions, exercises** and come with doubts. **You should be done with most of the problems of all the books under Arihant Series before you sit the entrance. Also, since most of the chapters are completed by now, start exploring previous year ISI, CMI, JEE advanced questions from now.**

2. $X_0, X_1, \alpha \in (0,1)$. $X_{n+1} = \alpha X_n + (1-\alpha)X_{n-1}\ \forall n \geq 1$. Prove that, $\{X_n\}$ converges and find the limit.

3. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ and f is monotonically increasing. Then, prove that $\exists a$ such that $f(x) = ax\ \forall x \in \mathbb{R}$.

4. **Algorithm to Compute $\sqrt{\alpha}, \alpha > 0$:** Start with any $x_1 > \sqrt{\alpha}$. Define $x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})$. Show that $\{x_n\}$ is monotonically decreasing and hence find the limit of $\{x_n\}$.

5. Consider the following 4 sets: $A = \{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\}, B = \{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\}, C = \{(x,y) : x^3 - 3xy^2 + 3y = 1\}, D = \{(x,y) : 3x^2y - 3x - y^3 = 0\}$. Prove that $A \cap B = C \cap D$.

6. $f(x) = k + \beta$ if $[x] = k \bmod 3$, where $k \in \{0,1,2\}$. If $\sum_{n=1}^{\infty} \frac{f(3^n\sqrt{2024})}{3^n} = 0$. Find the value of β.

7. Prove that for any prime $p \geq 3$, the number $\binom{2p-1}{p-1} - 1$ is divisible by p^2.

8. For sets A, B, let $f : A \to B$ and $g : B \to A$ be functions such that $f(g(x)) = x$ for each x. Prove that :
   - f need not be one one
   - f must be onto
   - g must be one one
   - g need not to be onto.

9. Let $x_n \to x$ and $y_n \to y$. Then, prove that $\frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} \to xy$.

10. Let $0 < a \leq x_1 \leq x_2 \leq b$. Define $x_n = \sqrt{x_{n-1}x_{n-2}}$ for $n \geq 3$. Show that $a \leq x_n \leq b$ and $|x_{n+1} - x_n| \leq \frac{b}{a+b}|x_n - x_{n-1}|$ for $n \geq 2$. Prove $\{x_n\}$ is convergent.

11. Let $0 < y_1 < x_1$. Define $x_{n+1} = \frac{x_n + y_n}{2}$ and $y_{n+1} = \sqrt{x_ny_n}$, for $n \in \mathbb{N}$. Prove that:
    - $\{y_n\}$ is increasing and bounded above while $\{x_n\}$ is decreasing and bounded below.
    - $0 < x_{n+1} - y_{n+1} < 2^{-n}(x_1 - y_1)$ for $n \in \mathbb{N}$.
    - Prove that x_n and y_n converge to the same limit.

12. **Euler's Constant:** Let
$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n t^{-1}dt$$
    - Show that γ_n is a decreasing sequence.
    - Show that $0 < \gamma_n \leq 1$ for all n.

- $lim \gamma_n$ exists and is denoted by γ. The real real number γ is called the Euler's constant.

13. Evaluate $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n.3^m+m.3^n)}$

14. Find the value of $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}{}_{i\neq j\neq k}\frac{1}{3^i3^j3^k}$, where the sum is taken over i, j, k such that no two of them can be equal.

15. Let $S_n, n = 1, 2, 3, \cdots$ be the sum of infinite geometric series, whose first term is n and the common ratio is $\frac{1}{n+1}$. Evaluate

$$lim_{n\to\infty}\frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \cdots + S_nS_1}{S_1^2 + S_2^2 + \cdots + S_n^2}$$

.

16. $t_n = \frac{n^5+n^3}{n^4+n^2+1}, S_r = \sum_{n=1}^{r} t_r$. Find an explicit expression of S_r.

Problem Set 30

Srijan Chattopadhyay

September 1, 2024

1. Find the following limits (if exists):
   - $\underset{x\to 0}{lim}\frac{(x-\sin x)}{x^3}$
   - $\underset{x\to 0}{lim}\frac{(e^x-1-x)}{x^2}$
   - $\underset{x\to 0}{lim}(\frac{a^x+b^x+c^x}{3})^{1/x}, a, b, c > 0$
   - $\underset{x\to 0}{lim}((x+a)(x+b)(x+c))^{1/3} - x$
2. $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \le \lambda|x - y|$ for all $x, y \in \mathbb{R}$ for some $\lambda > 0$. Prove that f is continuous (This is called **Lipschitz continuous**).
3. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \le \lambda(x - y)^2$ for all $x, y \in \mathbb{R}$ for some $\lambda > 0$.
4. $f(x) = x\sin(1/x)$ if $x \ne 0$ and $= 0$ if $x = 0$. Prove that x is continuous.
5. Can you find minimum value of r such that if $f(x) = x\sin(1/x)$ if $x \ne 0$ and $= 0$ if $x = 0$, then f is at least once differentiable?
6. $$f(x) = \left\{ \begin{array}{ll} 0, & \text{for } x \in Q \\ 1, & \text{for } x \in Q^c \end{array} \right\}$$
   Q is the set of all rational numbers. Find all continuity points of f (This function is called Dirichlet Function).
7. Can there exist any continuous $f : \mathbb{R} \to \mathbb{R}$ which is rational on irrational points and irrational on rational points?
8. $f : \mathbb{R} \to \mathbb{R}$ and $\exists\alpha \in (0, 1)$ such that $|f(x) - f(y)| \le \alpha|x - y|$. Prove that f has a unique fixed point. (A point x is called a fixed point of f is $f(x) = x$).
9. $f, g : [0, 1] \to [0, 1]$ are continuous functions such that $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$. Show that $\exists c \in [0, 1]$ such that $f(c) = g(c)$.
10. All Sessions and Exercises of Limits and Functions chapters of Calculus book of Arihant.

Problem Set 31

Srijan Chattopadhyay

September 16, 2024

1. How many integral solutions are there to $x + y + z + t = 29$, when $x \geq 1, y > 1, z \geq 3, t \geq 0$?

2. If there are l objects of one kind, m objects of 2nd kind, n objects of third kind and so on, prove that the number of ways of choosing r objects from the $l + m + n + \cdots$ objects is the coefficient of x^r in the expansion of $(1 + x + x^2 + \cdots + x^l)(1 + x + \cdots + x^m)(1 + x + \cdots + x^n) \cdots$

3. There are n straight lines in a plane such that n_1 of them are parallel in one direction, n_2 of them in different direction and so on, n_k are parallel in another direction such that $n_1 + \cdots + n_k = n$. Also, no three of the given lines meet at a point. Prove that the total number of points of intersection is $\frac{1}{2}(n^2 - \sum_{r=1}^{k} n_r^2)$.

4. Given $\alpha \in \mathbb{C}$, define a function ρ as $\rho(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ for $|z| \leq 1$. Assume $|\alpha| < 1$. Then,

   (a) Show that, $|z| = 1 \implies |\rho(z)| = 1$ and $|z| < 1 \implies |\rho(z)| < 1$.

   (b) Is ρ bijective? If so, what will be its inverse?

   (c) Show that $\exists$ a unique $Z_0 \in \mathbb{C}$ such that $|Z_0| \leq 1$ and $\rho(Z_0) = Z_0$

5. Suppose that a function f is continuous on the interval $[a, b]$ and differentiable on (a, b). If the graph of f is not a line segment, prove that there exists a point $c \in (a, b)$ such that $|f'(c)| > |\frac{f(b)-f(a)}{b-a}|$

Problem Set 32

Srijan Chattopadhyay

October 2, 2024

1. Let f be a twice differentiable function on the open interval $(-1, 1)$ such that $f(0) = 1$. Suppose f also satisfies $f(x) \geq 0, f'(x) \leq 0$ and $f''(x) \leq f(x)$, for all $x \geq 0$. Show that $f'(0) \geq -\sqrt{2}$.

2. Find the limit
$$\lim_{x \to 0} \frac{\sin \tan x - \tan \sin x}{\arcsin \arctan x - \arctan \arcsin x}$$

3. Calculate the 100th derivative of the function
$$\frac{x^2 + 1}{x^3 - x}$$

4. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection of the positive integers. Prove that at least one of the following limits is true:
$$\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n + f(n)} = \infty; \qquad \lim_{N \to \infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{f(n)} \right) = \infty.$$
(**Caution!!! Hard Problem, try after solving the rest easy problems!!**)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x) > f(x) > 0$ for all real numbers x. Show that $f(8) > 2024f(0)$.

6. Evaluate
$$\frac{1/2}{1 + \sqrt{2}} + \frac{1/4}{1 + \sqrt[4]{2}} + \frac{1/8}{1 + \sqrt[8]{2}} + \frac{1/16}{1 + \sqrt[16]{2}} + \cdots$$

7. Prove that for any function $f : \mathbb{Q} \to \mathbb{Z}$, there exist $a, b, c \in \mathbb{Q}$ such that $a < b < c$, $f(b) \geq f(a)$ and $f(b) \geq f(c)$.

8. Define the sequence $x_1, x_2, \ldots$ by the initial terms $x_1 = 2, x_2 = 4$, and the recurrence relation
$$x_{n+2} = 3x_{n+1} - 2x_n + \frac{2^n}{x_n} \quad \text{for} \quad n \geq 1.$$
Prove that $\lim_{n \to \infty} \frac{x_n}{2^n}$ exists and satisfies
$$\frac{1 + \sqrt{3}}{2} \leq \lim_{n \to \infty} \frac{x_n}{2^n} \leq \frac{3}{2}.$$
(**Caution!!! A bit on the harder side**)

9. Let $x_1 = 2021$, $x_n^2 - 2(x_n + 1)x_{n+1} + 2021 = 0$ $(n \geq 1)$. Prove that the sequence x_n converges. Find the limit $\lim_{n \to \infty} x_n$ (**Can be a bit tricky to handle**)

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Prove that
$$\left| f(1) - \int_0^1 f(x)dx \right| \leq \frac{1}{2} \max_{x \in [0,1]} |f'(x)|.$$
(**You can do it with existing knowledge of integration!**)

Problem Set 33

Srijan Chattopadhyay

October 21, 2024

1. Let $0 < a < 1$. Find all functions $f : \mathbb{R} \to \mathbb{R}$ continuous at $x = 0$ such that $f(x) + f(ax) = x, \forall x \in \mathbb{R}$.

2. How many ordered pairs of real numbers (a, b) satisfy equality $\lim_{x\to 0} \frac{\sin^2 x}{e^{ax}-2bx-1} = \frac{1}{2}$?

3. What is the largest amount of complex z solutions a system can have? $|z-1||z+1| = 1$ $Im(z) = b$? (where b is a real constant)

4. (*) For a given integer $n \geq 1$, let $f : [0, 1] \to \mathbb{R}$ be a non-decreasing function. Prove that
$$\int_0^1 f(x)dx \leq (n+1)\int_0^1 x^n f(x)dx.$$

5. For $n = 1, 2, \ldots$ let
$$S_n = \log\left(\sqrt[n^2]{1^1 \cdot 2^2 \cdot \ldots \cdot n^n}\right) - \log(\sqrt{n}),$$
where log denotes the natural logarithm. Find $\lim_{n\to\infty} S_n$.

6. Let V be the set of all continuous functions $f\colon [0, 1] \to \mathbb{R}$, differentiable on $(0, 1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0, 1)$ such that
$$f(\xi) + \alpha = f'(\xi)$$

Problem Set 34

Srijan Chattopadhyay

October 29, 2024

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function whose second derivative is continuous. Suppose that f and f'' are bounded. Show that f' is also bounded.

2. Let m, n be integers such that $n \geq m \geq 1$. Prove that $\frac{\gcd(m,n)}{n}\binom{n}{m}$ is an integer. Here gcd denotes greatest common divisor and $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ denotes the binomial coefficient.

3. Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n + 1$. Here $\mathbb{N}$ is the positive integers $\{1, 2, 3, \dots\}$.

4. For $n \in \mathbb{N}$, define $a_n = \frac{1+1/3+1/5+\cdots+1/(2n-1)}{n+1}$ and $b_n = \frac{1/2+1/4+1/6+\cdots+1/(2n)}{n}$. Find the maximum and minimum of $a_n - b_n$ for $1 \leq n \leq 999$.

5. Calculate the exact value of the series $\sum_{n=2}^{\infty} \log(n^3 + 1) - \log(n^3 - 1)$ and provide justification.

6. Give all possible representations of 2022 as a sum of at least two consecutive positive integers and prove that these are the only representations.

7. Find the limit
$$\lim_{n\to\infty} \left(\frac{\left(1 + \frac{1}{n}\right)^n}{e} \right)^n.$$

8. A real-valued function f defined in (a, b) is said to be convex if
$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$
whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.
   - Prove that every convex function is continuous.
   - Prove that every increasing convex function of a convex function is convex.
   - If f is convex in (a, b) and if $a < s < t < u < b$, show that
   $$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$
   - Let $f : (a, b) \to \mathbb{R}$ such that f'' exists on (a, b). Then f is convex iff $f''(x) > 0 \; \forall x \in (a, b)$.

9. Consider the following number: 2581452668046920778582615126 63. It is the 13th power of some natural number. Find the number.

10. Practice objectives from Arihant calculus book. Will take a class test only on objectives (UGA) of whole syllabus very soon.

Problem Set 35

Srijan Chattopadhyay

November 12, 2024

1. $f : [0,1] \to \mathbb{R}$ differentiable on $(0,1)$, such that $\exists a, b \in (0,1)$ such that $\int_0^a f(x)dx = 0$ and $\int_b^1 f(x)dx = 0$. Also, $|f'(x)| \leq M \ \forall x \in (0,1)$. Prove that $|\int_0^1 f(x)dx| \leq \frac{(1-a+b)M}{4}$.
    - First prove that, $|\int_0^1 f(x)dx| \leq \frac{M(1-a)}{2}$ [Hint: Use MVT and a substitution to use $\int_0^a f(x)dx = 0$, Think how you can transform the interval $(0,1)$ to $(0,a)$].
    - Then, prove that $|\int_0^1 f(x)dx| \leq \frac{Mb}{2}$ [Hint: Again use MVT and a substitution to use $\int_b^1 f(x)dx = 0$, Think how you can transform the interval $(0,1)$ to $(b,1)$]
2. Show that there doesn't exist any increasing differentiable function $f : \mathbb{R} \to \mathbb{R}^+$ such that $f(f(x)) = f'(x)$
3. Determine all continuous functions $f : [0,1] \to \mathbb{R}$ that satisfy $\int_0^1 f(x)(1 - f(x))dx = \frac{1}{12}$
4. Let $f(x) = \int_0^1 |t - x|tdt \ \forall x \in \mathbb{R}$. Sketch the graph of $f(x)$. What is the minimum value of $f(x)$.
5. $f(x) = \int_x^{x+1} \sin(u^2)du$, find $\lim\limits_{x \to \infty} f(x)$, if it exists.

Problem Set 36

Srijan Chattopadhyay

December 10, 2024

1. Find all positive integers a, b, c, d, and n satisfying $n^a + n^b + n^c = n^d$ and prove that these are the only such solutions.

2. Suppose A is a singular matrix of order 3 with complex entries all of which having absolute value 1. Show that two rows or two columns of the matrix A are proportional.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f^3(x) = x$. Prove that $f^2(x) = x$.

4. Find

   (a) $\lim_{n\to\infty} \dfrac{\gcd(1,6) + \gcd(2,6) + \cdots + \gcd(n,6)}{1 + 2 + \cdots + n}$

   (b) $\lim_{n\to\infty} \dfrac{\operatorname{lcm}(1,6) + \operatorname{lcm}(2,6) + \cdots + \operatorname{lcm}(n,6)}{1 + 2 + \cdots + n}$

5. The sequence $\{q_n(x)\}$ of polynomials is defined by

$$q_1(x) = 1 + x, \quad q_2(x) = 1 + 2x$$

   and for $m \geq 1$ by

$$q_{2m+1}(x) = q_{2m}(x) + (m+1)xq_{2m-1}(x),$$
$$q_{2m+2}(x) = q_{2m+1}(x) + (m+1)xq_{2m}(x).$$

   Let x_n be the largest real solution of $q_n(x) = 0$. Prove that

   (a) the sequence $\{x_n\}$ is increasing.

   (b) $x_{2m+2} > -\frac{1}{m+1}$ for $m \geq 1$.

   (c) the sequence $\{x_n\}$ converges to 0.

6. Let the positive integers a, b, c be such that $a \geq b \geq c$ and $(a^x - b^x - c^x)(x-2) > 0$ for all $x \neq 2$. Show that a, b, c are sides of a right-angled triangle.

7. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and

$$f(0) = 0, \quad f(1) = 1.$$

   (a) Show that there exists x_1 in $(0,1)$ such that

$$\frac{1}{f'(x_1)} = 1.$$

   (b) Show that there exist distinct x_1, x_2 in $(0,1)$ such that

$$\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2.$$

   (c) Show that for a positive integer n, there exist distinct $x_1, x_2, \ldots, x_n$ in $(0,1)$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.$$

8. Let $\mathcal{P}_n$ denote the collection of polynomials of degree n such that the polynomial and all its derivatives have integer roots.

   (a) Find a polynomial in $\mathcal{P}_2$ having at least two distinct roots.

   (b) Find a polynomial in $\mathcal{P}_3$ having at least two distinct roots.

   (c) For any polynomial P in $\mathcal{P}_n$, show that the arithmetic mean of all roots of P is also an integer.

9. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1, f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for n.

10. Find $\sum_{n=2}^{\infty} \frac{n^2-2n-4}{n^4+4n^2+16}$.

11. Define a sequence (a_n) for $n \geq 1$ by $a_1 = 2$ and $a_{n+1} = a_n^{1+n^{-3/2}}$. Is $\lim_{n\to\infty} a_n < \infty$?

12. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \ldots + a_1x + 1$ be a polynomial where the a_i $(1 \leq i \leq 97)$ are real numbers. Prove that the equation $P(x) = 0$ has at least one nonreal root.

13. Find $\lim_{x\to\infty} \left((2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x\right)$

14. Find $\sum_{k=1}^{\infty} \frac{k^2-2}{(k+2)!}$.

15. A sequence (a_n) is defined by $a_0 = -1, a_1 = 0$, and $a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$ for all positive integers n. Find a_{100}.

Problem Set 37

Srijan Chattopadhyay

December 16, 2024

1. Suppose that $f : [-1, 1] \to \mathbb{R}$ is continuous and satisfies
$$\left(\int_{-1}^{1} e^x f(x)dx\right)^2 \geq \left(\int_{-1}^{1} f(x)dx\right)\left(\int_{-1}^{1} e^{2x} f(x)dx\right).$$
Prove that there exists a point $c \in (-1, 1)$ such that $f(c) = 0$. [You can assume CS inequality for integrals.]

2. Let f and g be two continuous, distinct functions from $[0, 1] \to (0, +\infty)$ such that
$\int_0^1 f(x)dx = \int_0^1 g(x)dx$
Let
$y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)}dx$, for $n \geq 0$, natural.
Prove that (y_n) is an increasing and divergent sequence.

3. Determine all triples (k, m, n) of positive integers satisfying
$$k! + m! = k!n!.$$

4. Let $\omega \neq 1$ be a 13th root of unity. Find the remainder when
$$\prod_{k=0}^{12}\left(2 - 2\omega^k + \omega^{2k}\right)$$
is divided by 1000.

5. Real numbers x and y with $x, y > 1$ satisfy $\log_x(y^x) = \log_y(x^{4y}) = 10$. What is the value of xy?

6. Find the largest possible real part of
$$(75 + 117i)z + \frac{96 + 144i}{z}$$
where z is a complex number with $|z| = 4$.

Problem Set 38

Srijan Chattopadhyay

January 4, 2025

True False

1. Define the right derivative of a function f at $x = a$ to be the following limit if it exists:

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \quad \text{where } h \to 0^+ \text{ means } h \text{ approaches 0 through positive values.}$$

   Statements:

   (a) If f is differentiable at $x = a$, then f has a right derivative at $x = a$.

   (b) $f(x) = |x|$ has a right derivative at $x = 0$.

   (c) If f has a right derivative at $x = a$, then f is continuous at $x = a$.

   (d) If f is continuous at $x = a$, then f has a right derivative at $x = a$.

2. Suppose a rectangle $EBFD$ is given and a rhombus $ABCD$ is inscribed in it such that the point A is on side ED of the rectangle. The diagonals of $ABCD$ intersect at point G. See the indicative figure below.

   Statements:

   (a) Triangles $\triangle CGD$ and $\triangle DFB$ must be similar.

   (b) It must be true that $\frac{AC}{BD} = \frac{EB}{ED}$.

   (c) Triangle $\triangle CGD$ cannot be similar to triangle $\triangle AEB$.

   (d) For any given rectangle $EBFD$, a rhombus $ABCD$ as described above can be constructed.

3. This question is about complex numbers.
   Statements:

   (a) The complex number e^{3i} lies in the third quadrant.

   (b) If $|z_1| - |z_2| = |z_1 + z_2|$ for some complex numbers z_1 and z_2, then z_2 must be 0.

   (c) For distinct complex numbers z_1 and z_2, the equation $|(z - z_1)^2| = |(z - z_2)^2|$ has at most 4 solutions.

   (d) For each nonzero complex number z, there are more than 100 numbers w such that $w^{2023} = z$.

4. **Statements:**

   (a) $\lim_{x \to 0} e^{1/x} = +\infty$.

   (b) $\lim_{x \to \infty} \frac{\ln x}{x^{100}} < \lim_{x \to \infty} \frac{\ln x}{x^{1/100}}$.

   (c) For any positive integer n,

$$\int_{-n}^{n} x^{2023} \cos(nx)\, dx < \frac{n}{2023}.$$

   (d) There is no polynomial $p(x)$ for which there is a single line that is tangent to the graph of $p(x)$ at exactly 100 points.

5. **Statements:**

   (a) $4 < \sqrt{5+5\sqrt{5}}$.

   (b) $\log_2 11 < \frac{1+\log_2 61}{2}$.

   (c) $(2023)^{2023} < (2023!)^2$.

   (d) $92^{100} + 93^{100} < 94^{100}$.

6. For a sequence a_i of real numbers, we say that $\sum a_i$ converges if $\lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i\right)$ is finite. In this question, all $a_i > 0$.

   Statements:

   (a) If $\sum a_i$ converges, then $a_i \to 0$ as $i \to \infty$.

   (b) If $a_i < \frac{1}{i}$ for all i, then $\sum a_i$ converges.

   (c) If $\sum a_i$ converges, then $\sum (-1)^i a_i$ also converges.

   (d) If $\sum a_i$ does not converge, then $\sum_i \tan(a_i)$ cannot converge.

Problem Set 39

Srijan Chattopadhyay

January 5, 2025

UGA

1. The number of positive divisors of $2^{24} - 1$ is: (A) 192 (B) 48 (C) 96 (D) 24.
2. The equation $\text{Re}(z^2) = 0$ represents: (A) a circle (B) a pair of straight lines (C) an ellipse (D) a parabola.
3. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $\det(A^3) = 125$, then the values of α are: (A) ± 1 (B) ± 2 (C) ± 3 (D) ± 5.
4. Let A, B, C be three non-collinear points in a plane. The number of points at a distance 1 from A, 2 from B, and 3 from C is: (A) exactly 1 (B) at most 1 (C) at most 2 (D) always 0.
5. Let $A = \{x \in [-2, 3] : \cos x > 0\}$. Then: (A) $\inf A = 0$ (B) $\sup A = \pi$ (C) $\inf A = -\pi/2$ (D) $\sup A = 3$.
6. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \frac{2023}{n}|a_n - a_{n-1}|$, $\forall n$. Then the sequence $\{a_n\}$ is: (A) not Cauchy (B) Cauchy but not convergent (C) convergent (D) not bounded.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and F be a primitive of f (i.e., $F' = f$). If $3x^2F(x) = f(x)$ for all $x \in \mathbb{R}$, then $f(x) =$: (A) e^{x^3} (B) $3x^2e^{x^3}$ (C) $x^2e^{x^2}$ (D) $3xe^{x^3}$.
8. $1 \times 2 - 2 \times 3 + 3 \times 4 - 4 \times 5 + \cdots - 2022 \times 2023 =$: (A) $(-2)(1011)(1012)$ (B) $-(1011)(1012)$ (C) $(-4)(1011)(1012)$ (D) $2(1011)(1012)$.
9. The number of times the digit 7 is written while listing all integers from 1 to $100,000$ is: (A) 10^4 (B) $5(10)^4 - 1$ (C) 10^5 (D) $5(10)^4$.
10. The differential equation $y'^2 - (x + \sin x)y' + x \sin x = 0$, with $y(0) = 0$, has: (A) a unique solution (B) two solutions (C) no solution (D) four solutions.

UGB

1. Consider $f(x) = x\lfloor x^2 \rfloor$, where $\lfloor x^2 \rfloor$ is the greatest integer less than or equal to x^2. Find the area of the region above the X-axis and below $f(x)$ for $1 \leq x \leq 10$.
2. In how many ways can numbers from 1 to 100 be arranged in a circle such that the sum of each pair of integers placed opposite each other is the same? (Arrangements are equivalent up to rotation.)
3. Find all triplets (x, y, z) of integers satisfying:
$$x^2 + y^2 + z^2 = 16(x + y + z).$$
4. Suppose A is a singular matrix of order 3 with complex entries, all of which have absolute value 1. Show that two rows or two columns of the matrix A are proportional.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f^3(x) = x$. Prove that $f^2(x) = x$.

6. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 4$.

   (a) Find the determinant of the matrix:

   $$A = \begin{pmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{pmatrix}.$$

   (b) Find the maximum and minimum values of the determinant of A.

7. For every $t \in \mathbb{R}$, let L_t be the line segment joining $(0, 1)$ and $(t, 0)$. Suppose L_t intersects the parabola $y = x^2$ at the point P_t. Define $f : \mathbb{R} \to \mathbb{R}$ as the y-coordinate of P_t. Answer the following with justification:

   (a) Is $f(t)$ continuous?

   (b) Is $f(t)$ bounded?

   (c) Find $\lim_{t\to\infty} f(t)$.

   (d) Is $f(t)$ differentiable at $t = 0$?

8. The sequence $\{q_n(x)\}$ of polynomials is defined as follows:

   $$q_1(x) = 1 + x, \quad q_2(x) = 1 + 2x,$$
   $$q_{2m+1}(x) = q_{2m}(x) + (m+1)xq_{2m-1}(x),$$
   $$q_{2m+2}(x) = q_{2m+1}(x) + (m+1)xq_{2m}(x),$$

   for $m \geq 1$. Let x_n be the largest real root of $q_n(x) = 0$. Prove that:

   (a) The sequence $\{x_n\}$ is increasing.

   (b) $x_{2m+2} > -\frac{1}{m+1}$ for $m \geq 1$.

   (c) The sequence $\{x_n\}$ converges to 0.

Problem Set 40

Srijan Chattopadhyay

January 30, 2025

1. Let $A(t)$ denote the area bounded by the curve $y = e^{-|x|}$, the X-axis, and the straight lines $x = -t$, $x = t$, then $\lim_{t\to\infty} A(t)$ is
(A) 2 (B) 1 (C) $\frac{1}{2}$ (D) e

2. How many triples of real numbers (x, y, z) are common solutions of the equations $x + y = 2$, $xy - z^2 = 1$?
(A) 0 (B) 1 (C) 2 (D) infinitely many.

3. For non-negative integers x, y, the function $f(x, y)$ satisfies the relations $f(x, 0) = x$ and $f(x, y + 1) = f(f(x, y), y)$. Then which of the following is the largest?
(A) $f(10, 15)$ (B) $f(12, 13)$ (C) $f(13, 12)$ (D) $f(14, 11)$.

4. Suppose p, q, r, s are $1, 2, 3, 4$ in some order. Let
$$x = \frac{1}{p + \frac{1}{q + \frac{1}{r + \frac{1}{s}}}}$$
We choose p, q, r, s so that x is as large as possible, then s is
(A) 1 (B) 2 (C) 3 (D) 4.

5. Let
$$f(x) = \begin{cases} 3x + x^2 & x < 0 \\ x^3 + x^2 & x \geq 0 \end{cases}$$
Then $f''(0)$ is
(A) 0 (B) 2 (C) 3 (D) None of these.

6. There are 8 teams in the pro-kabaddi league. Each team plays against every other exactly once. Suppose every game results in a win, that is, there is no draw. Let $w_1, w_2, \ldots, w_8$ be the number of wins and $l_1, l_2, \ldots, l_8$ be the number of losses by teams $T_1, T_2, \ldots, T_8$, then
   (a) $w_1^2 + \cdots + w_8^2 = 49 + (l_1^2 + \cdots + l_8^2)$.
   (b) $w_1^2 + \cdots + w_8^2 = l_1^2 + \cdots + l_8^2$.
   (c) $w_1^2 + \cdots + w_8^2 = 49 - (l_1^2 + \cdots + l_8^2)$.
   (d) None of these.

7. The remainder when $m + n$ is divided by 12 is 8, and the remainder when $m - n$ is divided by 12 is 6. If $m > n$, then the remainder when mn is divided by 6 is:
   (a) 1
   (b) 2
   (c) 3
   (d) 4

8. Let

$$A = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \ddots & \vdots & \\ (n-1)n+1 & (n-1)n+2 & \dots & n^2 \end{bmatrix}.$$

Select any entry and call it x_1. Delete the row and column containing x_1 to get an $(n-1) \times (n-1)$ matrix. Then select any entry from the remaining entries and call it x_2. Delete the row and column containing x_2 to get an $(n-2) \times (n-2)$ matrix. Perform n such steps. Then $x_1 + x_2 + \cdots + x_n$ is:

   (a) n
   (b) $\frac{n(n+1)}{2}$
   (c) $\frac{n(n^2+1)}{2}$
   (d) None of these.

9. The maximum of the areas of the rectangles inscribed in the region bounded by the curve $y = 3 - x^2$ and the X-axis is:

   (a) 4
   (b) 1
   (c) 3
   (d) 2

10. How many factors of $2^5 3^6 5^2$ are perfect squares?

   (a) 24
   (b) 20
   (c) 30
   (d) 36

11. How many 15-digit palindromes are there in each of which the product of the non-zero digits is 36 and the sum of the digits is equal to 15?

12. Let H be a finite set of distinct positive integers none of which has a prime factor greater than 3. Show that the sum of the reciprocals of the elements of H is smaller than 3. Find two different such sets with sum of the reciprocals equal to 2.5.

13. Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. Find the sum of all entries of A^{2015}.

14. Let A be an $n \times n$ matrix with real entries such that each row sum is equal to one. Find the sum of all entries of A^{2015}.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) > 0$ for all $x > 0$.

16. Give an example of a function which is continuous on $[0, 1]$, differentiable on $(0, 1)$, and not differentiable at the end points. Justify.

17. There are some marbles in a bowl. A, B, and C take turns removing one or two marbles from the bowl, with A going first, then B, then C, then A again, and so on. The player who takes the last marble from the bowl is the loser, and the other two players are the winners. If the game starts with N marbles in the bowl, for what values of N can B and C work together and force A to lose?

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(0)$ exists. Suppose $\alpha_n \neq \beta_n$, $\forall n \in \mathbb{N}$ and both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. Define

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove that $\lim_{n\to\infty} D_n = f'(0)$ under EACH of the following conditions:

(a) $\alpha_n < 0 < \beta_n$, $\forall n \in \mathbb{N}$.

(b) $0 < \alpha_n < \beta_n$ and $\frac{\beta_n}{\beta_n - \alpha_n} \leq M$, $\forall n \in \mathbb{N}$, for some $M > 0$.

(c) $f'(x)$ exists and is continuous for all $x \in (-1, 1)$.

19. Let $f(x) = x^5$. For $x_1 > 0$, let $P_1 = (x_1, f(x_1))$. Draw a tangent at the point P_1 and let it meet the graph again at point P_2. Then draw a tangent at P_2 and so on. Show that the ratio

$$\frac{A(\triangle P_n P_{n+1} P_{n+2})}{A(\triangle P_{n+1} P_{n+2} P_{n+3})}$$

is constant.

20. Let $p(x)$ be a polynomial with positive integer coefficients. You can ask the question: What is $p(n)$ for any positive integer n? What is the minimum number of questions to be asked to determine $p(x)$ completely? Justify.

Problem Set 41

Srijan Chattopadhyay

March 3, 2025

1. Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\bar{w}z}\right| < 1$$

if $|z| < 1$ and $|w| < 1$ and also

$$\left|\frac{w-z}{1-\bar{w}z}\right| = 1$$

if $|z| = 1$ and $|w| = 1$.

2. Prove that for a fixed w in the unit disc D, the mapping

$$F : z \to \left|\frac{w-z}{1-\bar{w}z}\right|$$

satisfies the following :

- F maps the unit disc to itself.
- F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.
- $|F(z)| = 1$ if $|z| = 1$.
- $F : D \to D$ is bijective.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be such that $f(z) = e^z$. Prove that f is neither one-to-one nor onto. Find a subset S on which f is one-to-one. Can you make this subset large so that it still remains one-to-one?

Problem Set 42

Srijan Chattopadhyay

March 12, 2025

1. Consider a function $f : [0,1] \to [0,1]$ satisfying the following property $|f(x) - f(y)| < |x - y|$ for all $x, y \in X, x \neq y$. Show that f has a fixed point. Is the fixed point unique? [Hint : Define $d(x) = |x - f(x)|$. Suppose $\inf_x d(x) \geq \epsilon > 0$. Assume $\exists x_0$ such that the infimum is attained. (Can you prove this? You can skip if you can't!). Then, use the property of f to arrive at a contradiction.]

2. Let $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n+1)$ Show that the polynomial $p(x)$ has no real root.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by
$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 5x - 6 & \text{if } x \notin \mathbb{Q}. \end{cases}$$
Find continuity points of f.

4. Show that the polynomial equation with real coefficients $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_3 x^3 + x^2 + x + 1 = 0$ cannot have all real roots.

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous, one-to-one function. If there exists a positive integer n such that $f^n(x) = x$, for every $x \in \mathbb{R}$, then prove that either $f(x) = x$ or $f^2(x) = x$. (Note that $f^n(x) = f(f^{n-1}(x))$.)

6. Consider $f(x) = \frac{x^3}{6} + \frac{x^2}{2} + \frac{x}{3} + 1$. Prove that $f(x)$ is an integer whenever x is an integer. Determine with justification, conditions on real numbers a, b, c and d so that $ax^3 + bx^2 + cx + d$ is an integer for all integers x.

7. Let A and B be finite subsets of the set of integers. Show that
$$|A + B| \geq |A| + |B| - 1.$$
When does equality hold? (Here $A + B = \{x + y : x \in A,\, y \in B\}$. Also, $|S|$ denotes the number of elements in the set S.)

Problem Set 43

Srijan Chattopadhyay

March 26, 2025

1. (a) Show that there does not exist a function $f : (0, \infty) \to (0, \infty)$ such that

$$f''(x) \leq 0 \text{ for all } x \text{ and } f'(x_0) < 0 \text{ for some } x_0.$$

(b) Let $k \geq 2$ be any integer. Show that there does not exist an infinitely differentiable function $f : (0, \infty) \to (0, \infty)$ such that

$$f^{(k)}(x) \leq 0 \text{ for all } x \text{ and } f^{(k-1)}(x_0) < 0 \text{ for some } x_0.$$

Here, $f^{(k)}$ denotes the k^{th} derivative of f.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Can you put appropriate conditions on f, so that f restricted to $(a, b), -\infty < a < b < \infty$ can't attain it's maximum and minimum inside (a, b)?

3. Let $a, b \in \mathbb{Z}$ and $b > 0$. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that

(a) f is continuous if and only if $a > 0$.
(b) $f'(0)$ exists if and only if $a > 1$.
(c) f' is bounded if and only if $a \geq 1 + b$.
(d) f' is continuous if and only if $a > 1 + b$.

4. Let $f : (0, 1] \to \mathbb{R}$ be a differentiable function with f' bounded on $(0, 1]$. Define

$$a_n = f\left(\frac{1}{n}\right), \quad n \geq 1.$$

Show that $\{a_n\}$ is a convergent sequence.

5. Let f be a thrice differentiable function on $(0, 1)$ such that $f(x) \geq 0$ for all $x \in (0, 1)$. If $f(x) = 0$ for at least two values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$.

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that $f(x + 1) - f(x)$ converges to 0 as $x \to \infty$. Then show that

$$\frac{f(x)}{x} \to 0 \quad \text{as } x \to \infty.$$

Problem Set 44

Srijan Chattopadhyay

March 29, 2025

1. A rectangle $HOME$ has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and E the foot of the altitude from A. What is the length of BC?

2. Let $a_1, \ldots, a_n \geq 0$, not all zero. Show that the equation
$$x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n = 0$$
has a unique positive real root. Moreover, if r be that root, then show that $r^b \geq a^a$ where $a = \sum_{j=1}^n a_j$ and $b = \sum_{j=1}^n j a_j$.

3. Let $f : [0, 1] \to \mathbb{R}$ be a real-valued continuous function which is differentiable on (0, 1) and satisfies $f(0) = 0$. Suppose that there exists a constant $c \in (0, 1)$ such that
$$|f'(x)| \leq c|f(x)| \text{ for every } x \in (0, 1).$$
Show that $f(x) = 0$ for all $x \in [0, 1]$.

4. Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that
$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called full if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k. How many full sequences of length n are there?

6. Let a, b be real roots in $(0, \frac{1}{2})$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying
$$f(f(x)) = af(x) + bx \text{ for every } x \in \mathbb{R}.$$
Show that $f(0) = 0$.

7. If α be a root of the equation
$$z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$$
where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$, then show that
$$|\alpha| \leq 1 + \max_{0 \leq k \leq n-1} |a_k|.$$

8. Let ABC be a triangle with incircle w touching BC at D, let DX be a diameter of the incircle w. If $\angle BXC = 90^\circ$, show that
$$5a = 3(b + c)$$
where a, b, c are lengths of BC, CA and AB, respectively.

9. Suppose that S is the set of all 100-digit natural numbers that are formed using the digits $1, 2, \ldots, 7$ only. For $n \in \mathbb{N}$, let $g(n)$ denote the product of the digits of n. Determine

$$\sum_{n \in S} g(n).$$

10. Let f be continuous, non-negative, and assume that

$$\int_0^\infty f(x)\,dx < \infty.$$

Then show that

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n x f(x)\,dx = 0.$$

Problem Set 45

Srijan Chattopadhyay

April 3, 2025

1. $f : [0, 1] \to \mathbb{R}$ is such that $f(x) = 1/2$ on $[0, 1/2)$ and $f(x) = 1$ on $[1/2, 1]$. In particular, f is not continuous at $1/2$. (You possibly know that a function which is discontinuous at finitely (in fact, countably) many points is riemann integrable?) Find the Riemann Integral of f. Call that $R(f)$. Now define
$$L(f) = \frac{1}{2} l(f^{-1}(1/2)) + 1.l(f^{-1}(1))$$
where $l(I)$ is the length of the interval I and $f^{-1}(c)$ is the set which maps to c by f. Prove that $L(f) = R(f)$. In fact, you can create such step functions which are discrete at finitely many points, show that for those also the notions of integrals match.

   Note: At this step, you can't go beyond this. But this concept of length (l) can be generalized for almost all sets in $\mathbb{R}$. Then you don't need a step function anymore to define L in a similar way. In fact it can be proved that almost all functions can be approximated pointwise by functions which take only finitely many values. Proof of that is not difficult and hence given in CT10, some of you even proved that in the exam as well. Then $L(f)$ is defined in terms of those approximating functions that take finitely many values in the range.

2. Prove that there exist 100 consecutive natural numbers such that exactly 3 of them are prime. [Hint: Use Discrete IVT]

3. If $f : [0, 1] \to \mathbb{R}$ is a continuous function, prove that $\lim_{n\to\infty} n \left(\int_0^1 f(x)x^n dx \right) = f(1)$.

4. (**From CT3**) Let p be a prime and m be a positive integer for which $m < p$ and suppose decimal expansion of $\frac{m}{p}$ has period $2k$ for some positive integer k.
$$\frac{m}{p} = 0.ABABABABABABAB\cdots$$
where A, B are 2 distinct blocks of k digits, prove that $A + B = 10^k - 1$.

5. (**From CT5**) Determine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ that satisfy
$$f(x) = (x + 1)f(x^2),$$
for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

6. (**From CT5**) Consider the set of integers $\{1, 2, ..., 100\}$. Let $\{x_1, x_2, ..., x_{100}\}$ be some arbitrary arrangement of the integers $\{1, 2, ..., 100\}$, where all of the x_i are different. Find the smallest possible value of the sum $S = |x_2 - x_1| + |x_3 - x_2| + ... + |x_{100} - x_{99}| + |x_1 - x_{100}|$.

7. (**From CT6**) Recall that the decimal expansion of a real number x is $I.d_1d_2\cdots$ if $x = I + \sum_{n=1}^{\infty} 10^{-n} d_n$ for some $I \in \mathbb{Z}$ and $d_1, d_2 \cdots \in \{0, 1, \cdots, 9\}$. Prove that for x, the decimal expansion $I.d_1d_2\cdots$ is not unique if and only if either $d_n = 0$ for all but finitely many n's or $d_n = 9$ for all but finitely many n's.

8. (**From CT6**) For any positive integer n, define $f(n)$ to be the smallest positive integer that does not divide n. For example, $f(1) = 2$, $f(6) = 4$. Prove that for any positive integer n, either $f(f(n))$ or $f(f(f(n)))$ must be equal to 2.

9. (**From CT6**) For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \ldots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.